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ION THERMAL CONDUCTIVITY AND ION DISTRIBUTION FUNCTION IN THE BANANA REGIME

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RESEARCH REPORT

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ABSTRACT

A new approximate collision operator which is more general than the model operator derived by HIRSHMAN and SIGMAR is presented. By use of this collision operator, the ion thermal conductivity and the ion distribution function in the banana regime are calculated for an axisymmetric toroidal plasma of arbitrary aspect ratio. In the limits of large and small aspect ratios , the computed thermal conductivity agrees with the results of ROSENBLUTH et al. and HAZELTINE et al. The simple expression for this conductivity is also derived.

The neoclassical transport coefficients in the low collision frequency regime have been investigated by many authors (HINTON and HAZELTINE,1976, HIRSHMAN and SIGMAR,1981). Using the large aspect ratio expansion, ROSENBLUTH et al.(1972) obtained the transport coefficients from a variational principle. The aspect ratios for the present and the planned future thermonuclear reactors are not so large that the finite aspect ratio correction is important. Recently, BOLTON and WARE (1983) calculated the ion thermal conductivity without assuming the large aspect ratio. They solved the drift kinetic equation numerically by using the model collision operator derived by HIRSHMAN and SIGMAR (1976). The finite aspect ratio correction is shown to increase the ion thermal conductivity by a factor of two in the intermidiate aspect ratio case. In this paper, we introduce a new approximate collision operator. Using this collision operator, we calculate the ion thermal conductivity and the ion distribution function in the banana regime for an axisymmetric toroidal plasma of arbitrary aspect ratio.

Let us introduce the flux coordinates (ψ,φ,χ) where ψ is the poloidal flux, φ the toroidal angle,and χ the poloidal angle-like variable. We consider a plasma composed of electrons and a single kind of ions in an axisymmetric magnetic field

$$\mathbf{B} = I\nabla\varphi + \nabla\varphi\times\nabla\psi, \tag{1}$$

where I is a function of ψ only. The mass and the charge of an electron are denoted by m_e and $-e$, and those of an ion by m_i and e_i.

The basic equation to describe the neoclassical transport is a drift kinetic equation for a guiding-center distribution function. In many situation of practical interest, the guiding-center distribution function f_i for ions is slightly perturbed from the local Maxwellian distribution function f_{i0}. We assume that the perturbation f_{i1} due to the ion-temperature gradient from f_{i0} is the first order quantity in a parameter $\delta=\rho_{ip}/L$, where ρ_{ip} is the poloidal Larmor radius of an ion and L is the plasma scale length. Then, the distribution function f_{i1} satisfies the linearized drift kinetic equation

$$v_{\parallel}\frac{B_{\chi}}{B}\frac{\partial f_{i1}}{\partial\chi} - C_{ii}(f_{i1}) = -\mathbf{V}_{di}\cdot\nabla\psi(x^2 - 5/2)\frac{T_i'}{T_i}f_{i0}, \tag{2}$$

where we have chosen the energy, the magnetic moment $\mu = v_\perp^2/2B$ and $\sigma = v_{\shortparallel}/|v_{\shortparallel}|$ as independent variables instead of the velocity. Here, $v_{\shortparallel} = \mathbf{v} \cdot \mathbf{B}/|\mathbf{B}|, x = v/v_i, v_i = \sqrt{2T_i/m_i}, T_i' = dT/d\psi$, T_i the temperature of ions, $\mathbf{V}_{di} \cdot \nabla\psi = (Im_i c/e_i)v_{\shortparallel} \cdot \nabla(v_{\shortparallel}/B)$ the radial component of drift velocity and C_{ii} the linearized Fokker-Planck operator for collisions between ions. We have neglected collisions with electrons in the above equation (2) since the mass ratio m_e/m_i is very small.

In the banana regime, we expand the distribution function as $f_{i1} = f_i^{(0)} + v_{ii} f_i^{(1)} + \cdots$, where $v_{ii} = 4\pi n_i e_i^4 \ln\Lambda/m_i^2 v_i^3$ is the ion-ion collision frequency. Then, the function $f_i^{(0)}$ and $f_i^{(1)}$ satisfy the equations

$$v_{\shortparallel} \frac{B_x}{B} \frac{\partial f_i^{(0)}}{\partial \chi} = -\mathbf{V}_{di} \cdot \nabla\psi (x^2 - 5/2) \frac{T_i'}{T_i} f_{i0}, \tag{3}$$

$$v_{\shortparallel} \frac{B_x}{B} \frac{\partial f_i^{(1)}}{\partial \chi} - C_{ii}(f_i^{(0)}) = 0. \tag{4}$$

The solution to the eq.(3) is given by

$$f_i^{(0)} = -\frac{m_i c I v_i}{e_i} \left[\frac{xq}{B}(x^2 - 5/2)f_{i0} + G(x,\lambda,\sigma) \right] \frac{T_i'}{T_i}, \tag{5}$$

where $q = v_{\shortparallel}/v, \lambda = (1-q^2)/B$. The function G is obtained from the following solvability condition for the eq.(4):

$$\int \frac{d\chi}{v_{\shortparallel}} \frac{B}{B_x} C_{ii}(\frac{xq}{B}(x^2 - 5/2)f_{i0} + G) = 0. \tag{6}$$

Here the integration means

$$\int \frac{d\chi}{v_{\shortparallel}} = \begin{cases} \int_0^{2\pi} \frac{d\chi}{v_{\shortparallel}} & \text{for } \lambda < \lambda_c, \\ \sum_\sigma \int_{\chi_1(\lambda)}^{\chi_2(\lambda)} \frac{d\chi}{|v_{\shortparallel}|} & \text{for } \lambda > \lambda_c. \end{cases} \tag{7}$$

where $\lambda_c = 1/B_{max}$, and $\chi_1(\lambda)$ and $\chi_2(\lambda)$ are bounce points of trapped particle. As is well known (ROSENBLUTH et al.,1972), the function G becomes identically zero in the trapped particle region.

The heat flux across a magnetic surface is given by

$$Q_i = <\int d\mathbf{v} \left(\frac{1}{2}m_i v^2 - \frac{5}{2}T_i\right) \mathbf{V}_{di} \cdot \nabla\psi f_{i1}>$$

$$= -\frac{m_i c I v_i T_i}{e_i} < \int d\mathbf{v} \frac{qx}{B}(x^2 - \frac{5}{2})C_{ii}(f_i^{(0)}) >. \tag{8}$$

Here, $<A>$ denotes the average over the magnetic surface and is expressed as

$$<A> = \int_0^{2\pi} d\chi (A/B_\chi) / \int_0^{2\pi} d\chi /B_\chi. \tag{9}$$

Let us introduce an approximate collision operator. The Legendre polynomial $P_l(q)$ is the eigenfunction of the collision operator C_{ii} (ROSENBLUTH et al.,1957),i.e.,

$$C_{ii}(\varphi(x)P_l(q)) = P_l(q)C_{ii}^l(\varphi(x)). \tag{10}$$

The collision operator C_{ii}^l is well approximated for $l^2 \gg 1$ by the pitch-angle scattering term,i.e.,

$$C_{ii}^l \cong - \frac{1}{2}l(l+1)\nu_{ii}(x), \tag{11}$$

where $\nu_{ii}(x) = (\nu_{ii}/2x^3)[(2x^2-1)\mathrm{erf}(x) + (2/\sqrt{\pi})xe^{-x^2}]$. We use this approximation for $l \geq 3$. Then, the collision operator is written as

$$C_{ii}(f) = \nu_{ii}(x)\mathcal{L}(f) + \sum_{l=0}^{2} P_l(q)[C_{ii}^l + \frac{l(l+1)}{2}\nu_{ii}(x)]f_l. \tag{12}$$

Here,

$$f_l = \frac{2l+1}{2}\int_{-1}^{1} dq P_l(q)f \tag{13}$$

and

$$\mathcal{L} = \frac{\nu_\parallel}{B}\frac{\partial}{\partial\mu}\mu\nu_\parallel\frac{\partial}{\partial\mu}. \tag{14}$$

This approximate collision operator preserves the properties of the particle number, the momentum and the energy conservations. Our collision operator (12) is more general than that of Hirshman and Sigmar which can be derived by approximating $C_{ii}^l(l=0,1,2)$ in (12) by more simple forms.

Let us define the function $K(x)$ by

$$K(x) = \frac{3}{2}<B^2>\int_0^{\lambda_c} d\lambda G(x,\lambda,\sigma=1). \tag{15}$$

Using the eq.(6) and the approximate collision operator (12), we can express the function G in terms of $K(x)$ as

$$G = \frac{1}{2}\theta(\lambda_c-\lambda)\sigma\int_\lambda^{\lambda_c}\frac{d\lambda}{<\sqrt{1-\lambda B}>}$$

$$\times \frac{1}{\nu_{ii}(x)}[C_{ii}^1(x(x^2-5/2)f_{i0}) + (C_{ii}^1 + \nu_{ii}(x))K(x)], \tag{16}$$

where $\theta(\lambda_c - \lambda)$ is the Heaviside function. The function $K(x)$ satisfies the equation

$$\nu_{ii}(x)K(x) - f_c(C_{ii}^1 + \nu_{ii}(x))K(x) = f_cC_{ii}^1(x(x^2-5/2)f_{i0}), \tag{17}$$

where

$$f_c = \frac{3}{4}<B^2>\int_0^{\lambda_c}\frac{\lambda d\lambda}{<\sqrt{1-\lambda B}>}. \tag{18}$$

This equation is derived from (15) and (16). Using the expression (16), we obtain

$$f_i^{(0)} = -\frac{m_iclv_i}{e_i}\frac{T_i'}{T_i}\left[\frac{1}{B}xq(x^2-\frac{5}{2})f_{i0}+\theta(\lambda_c-\lambda)\sigma\frac{1}{2f_c}\int_\lambda^{\lambda_c}\frac{d\lambda}{<\sqrt{1-\lambda B}>}K(x)\right]. \tag{19}$$

The heat flux Q_i is also written in terms of $K(x)$ as

$$Q_i = -n_i\left(\frac{m_icv_iI}{e_i<B>}\right)^2\frac{\sqrt{2}}{\tau_i}T_i'L_i, \tag{20}$$

$$L_i = -<\frac{1}{B^2}><B^2>\int_0^\infty dx x^3(x^2 - 5/2)\bar{C}_{ii}^1(x(x^2 - 5/2)e^{-x^2})$$

$$+\frac{<B>^2}{<B^2>}\int_0^\infty dx x^3(5/2 - x^2)\bar{C}_{ii}^1(\hat{K}(x)e^{-x^2}), \tag{21}$$

where $\hat{K}(x)=K(x)/f_{i0}, \bar{C}_{ii}^1 = C_{ii}^1/\nu_{ii}$, and $\tau_i = 3\sqrt{2\pi}/4\nu_{ii}$.

We calculate the function $K(x)$ by means of the Chapman-Enskog method. Let us expand

the function $K(x)$ as $K(x)=x\sum_{j=0}^{N-1}a_jS_{3/2}^{(j)}(x^2)f_{i0}$,where $S_{3/2}^{(j)}$ is the Sonine polynomial of order 3/2.

By substituting this expansion into (17), we obtain the following equation for a_j:

$$\sum_{k=0}^{N-1}\left[(1-f_c)<j|\bar{\nu}_{ii}(x)|k> - f_c<j|\bar{C}_{ii}^1|k>\right]a_k = - f_c<j|\bar{C}_{ii}^1|1>, \quad j=0,1,2,...N-1 \tag{22}$$

where $\bar{\nu}_{ii}(x)=\nu_{ii}(x)/\nu_{ii}$. Here

$$<j|C|j'> = \int_0^\infty dx x^3 S_{3/2}^{(j)}(x^2)C(xS_{3/2}^{(j')}(x^2)e^{-x^2}), \tag{23}$$

where $C=\bar{C}_{ii}^1$ or $\bar{\nu}_{ii}(x)$. The matrix elements $<j|C|j'>$ are calculated by using the generating

functions (KANEKO,1960, TAGUCHI,1982,1983).

In the limits of unit aspect ratio and large aspect ratio, the ion thermal conductivity can be

obtained analytically. The thermal conductivity for the unit aspect ratio ($f_c = 0$) becomes

$L_i = <B>^2<1/B^2>/\sqrt{2}$ since $K(x)=0$. This result agrees with the one of HAZELTINE et

al.(1973). The function $\hat{K}(x)$ for the uniform magnetic field ($f_c = 1$) is given by

$$\hat{K}(x) = \left[x(\frac{5}{2} - x^2) - \frac{<0|\bar{v}_{ii}(x)|1>}{<0|\bar{v}_{ii}(x)|0>} x \right]. \tag{24}$$

By means of the relation

$$\int_0^\infty dx e^{-x^2} x^3 (5/2 - x^2) \bar{C}_{ii}^1(\hat{K}) = <1|\bar{C}_{ii}^1|1>$$
$$+ (1 - f_c)/f_c \int_0^\infty dx\, e^{-x^2} x^3 (5/2 - x^2) \bar{v}_{ii}(x) \hat{K}, \tag{25}$$

we can obtain the thermal conductivity in the large aspect ratio

$$L_i = \frac{<B>^2}{\sqrt{2}} \left(<\frac{1}{B^2}> - \frac{1}{<B^2>} \right) + (1 - f_c)\frac{<B>^2}{<B^2>}\alpha \cong (1 - f_c)\alpha, \tag{26}$$

$$\alpha = -\frac{<0|\bar{v}_{ii}(x)|1>^2}{<0|\bar{v}_{ii}(x)|0>} + <1|\bar{v}_{ii}(x)|1> \cong 0.326. \tag{27}$$

This agrees with the result of ROSENBLUTH et al.(1972).

Next, we derive the simple expression for the ion thermal conductivity. The lowest approximation for $K(x)$ is given by $N=2$. Then, the approximate thermal conductivity becomes

$$L_i^{(ap)} = \frac{<B>^2}{\sqrt{2}} \left(<\frac{1}{B^2}> - \frac{1}{<B^2>} \right) + \frac{1}{\sqrt{2}}\frac{<B>^2}{<B^2>}\frac{1 - f_c}{1 + \beta f_c} \tag{28}$$

$$\beta = -1 + \frac{1}{\sqrt{2}\alpha} \cong 1.17. \tag{29}$$

As will be shown later, this is in very close agreement with the more rigorous one obtained by $N=50$. Note that the approximate thermal conductivity $L_i^{(ap)}$ for the limits of unit aspect ratio and large aspect ratio agrees with the exact one L_i.

Finally, we practically calculate the thermal conductivity L_i and the ion distribution function $f_i^{(0)}$ in a typical magnetic field of tokamak with circular cross section , i.e. ,

$$\mathbf{B} = (B_r, B_\zeta, B_\theta) = (0, B_{\zeta 0}/h, B_{\theta 0}/h), \tag{30}$$
$$h = 1 + \epsilon\cos\theta, \quad \epsilon = r/R. \tag{31}$$

Here r and θ are the polar coordinates in a plane perpendicular to the magnetic axis and R is the major radius. For this magnetic field, f_c is computed numerically and is shown in Fig.1. The dashed curve represents the analytic expression $f_c = 1 - 1.035\sqrt{2\epsilon}$ for small ϵ. In Fig.2, we depict the thermal conductivities $\bar{L}_i = \sqrt{2}L_i/\sqrt{\epsilon}$ and $\bar{L}^{ap} = \sqrt{2}L_i^{ap}/\sqrt{\epsilon}$ as a function of ϵ.

From this figure, the approximate thermal conductivity $\bar{L}_i^{ap}$ is shown to agree well with the more rigorous one $\bar{L}_i$. CHANG and HINTON (1982) derived the following approximate expression,

$$\bar{L}_i = (0.66 + 1.88\sqrt{\epsilon} - 1.54\epsilon)<B>^2<1/B^2> \qquad (32)$$

This expression was obtained by using the BOLTON and WARE's result for $\epsilon = 0.3$, and the results of ROSENBLUTH et al. for the large aspect ratio and of HAZELTINE et al. for the unit aspect ratio. For comparision, we also plot the values of this expression in Fig.2 by dashed curve. The ion distribution function $f_c^{(0)}$ is easily computed from (19). In Figs.3(a)-(d), we shows the contours of $f_c^{(0)}/f_{i0}$ for $\theta = 0$, and (a) $\epsilon = 0.1$, (b) $\epsilon = 0.3$, (c) $\epsilon = 0.5$. The contour levels are evenly spaced. The dashed lines represent the boundaries between the trapped and the untrapped region. In Fig.4, the distribution functions $\tilde{f}_i^{(0)} = f_i^{(0)}/f_{i0}(-m_i c l v_i T_i'/e_i T_i)^{-1}$ are plotted as a function of $v_{\parallel}/v_i$ for $\theta = 0$, $v_{\perp} = 0$, and $\epsilon = 0, 0.1, 0.3$ and 0.5.

In conclusion, we have presented the new approximate collision operator. Using this operator, we have formulated the method for calculating the ion thermal conductivity and the ion distribution function in the banana regime. Our method can be used for the axisymmetric toroidal plasma of arbitrary aspect ratio.

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FIGURE CAPTIONS

Fig.1   Values of f_c as a function of ϵ.

Fig.2   The ion thermal conductivities $\bar{L}_i$ and $\bar{L}_i^{ap}$ as a function of ϵ. The dashed curve shows the expression (32) obtained by CHANG and HINTON.

Fig.3   The contours of $f_i^{(0)}/f_{i0}$ for $\theta = 0$ and the inverse aspect ratios (a) $\epsilon = 0.1$, (b) $\epsilon = 0.3$, (c) $\epsilon = 0.5$. The contour levels are evenly spaced. The dashed lines represent the boundaries between the trapped and the untrapped region.

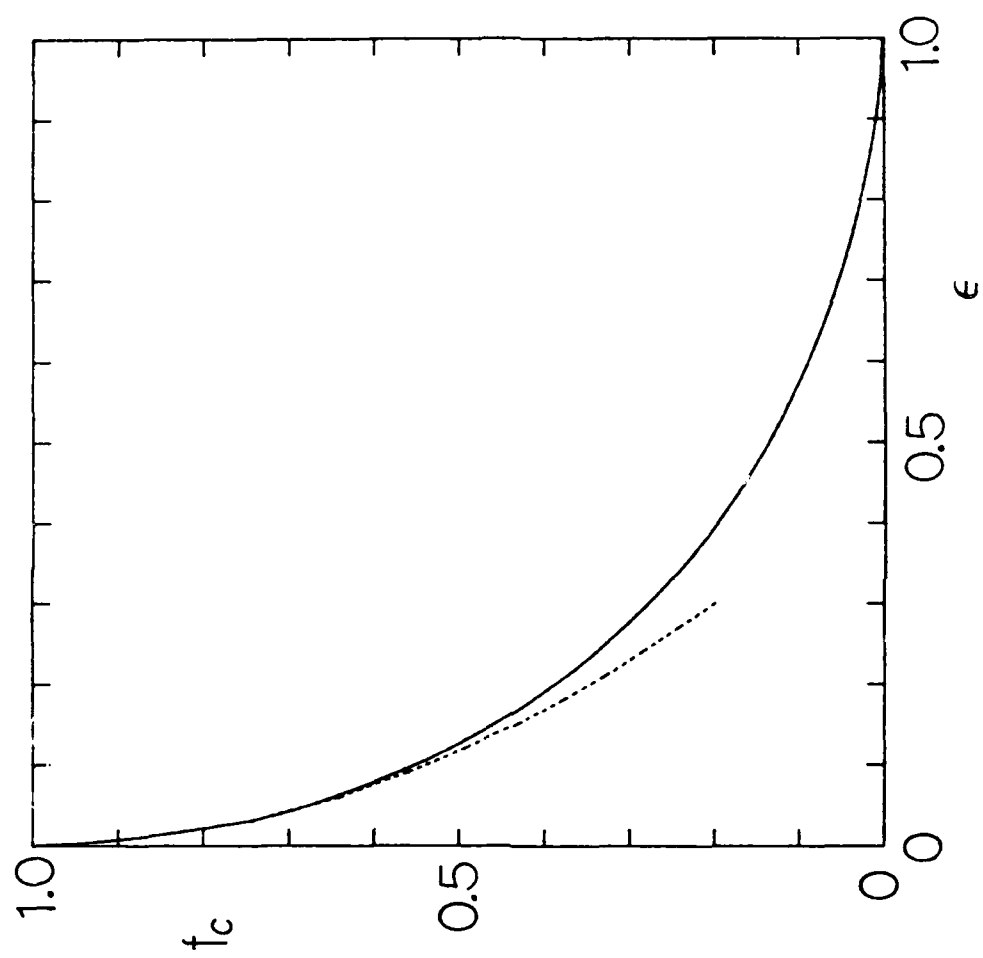Fig.4   The ion distribution functions $\tilde{f}_i^{(0)}$ are plotted as a function of $v_{\parallel}$ for $\theta = 0$, $v_{\perp} = 0$, and $\epsilon = 0, 0.1, 0.3$ and $.5$.
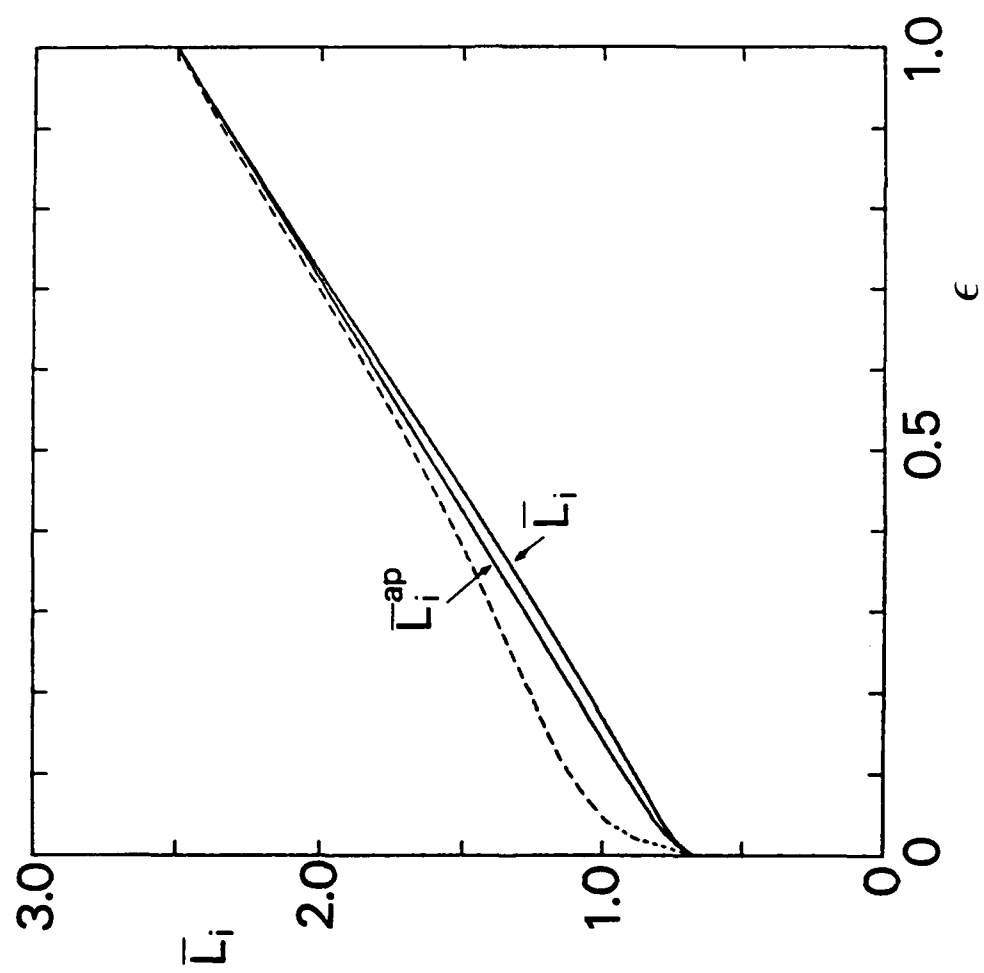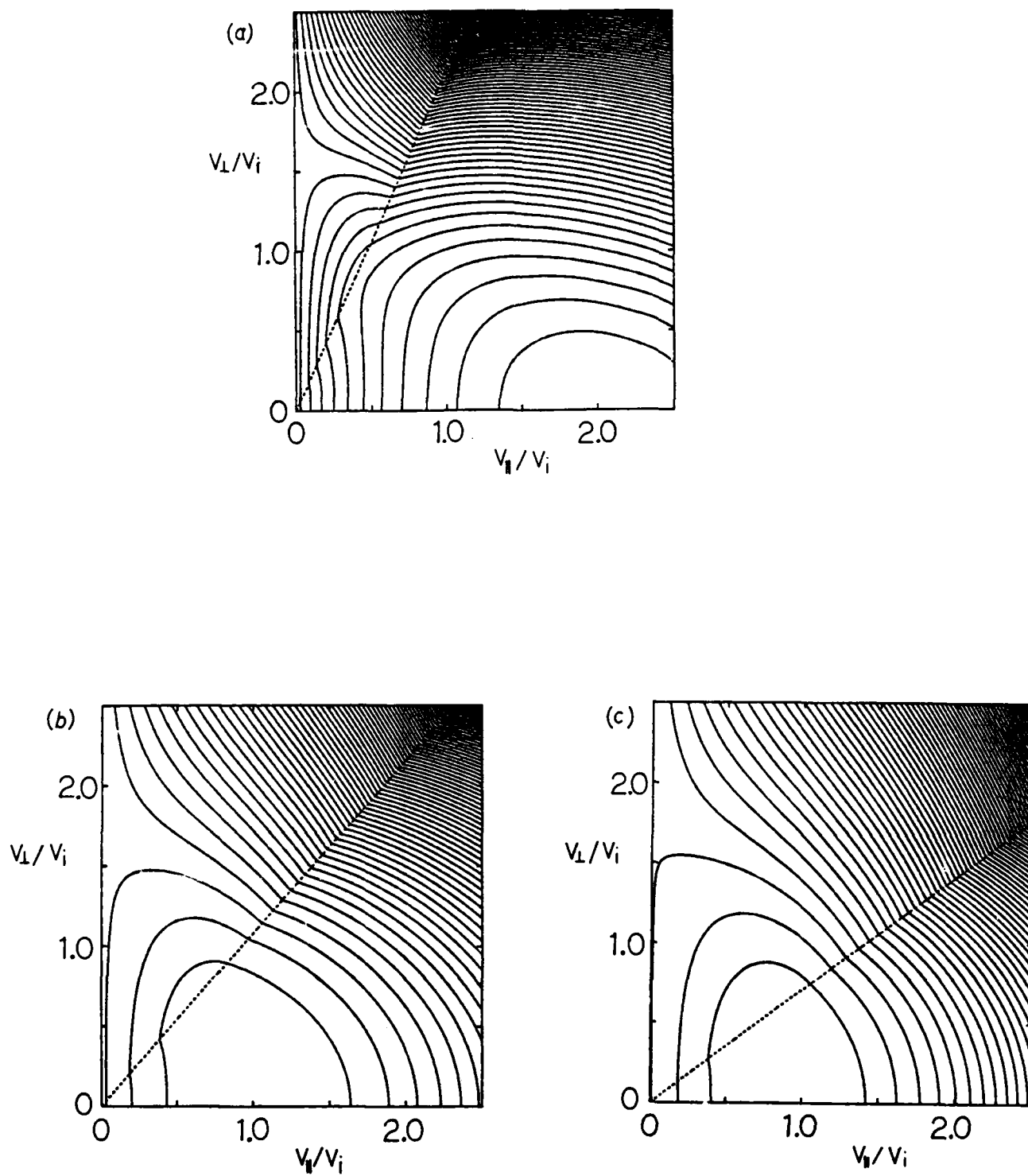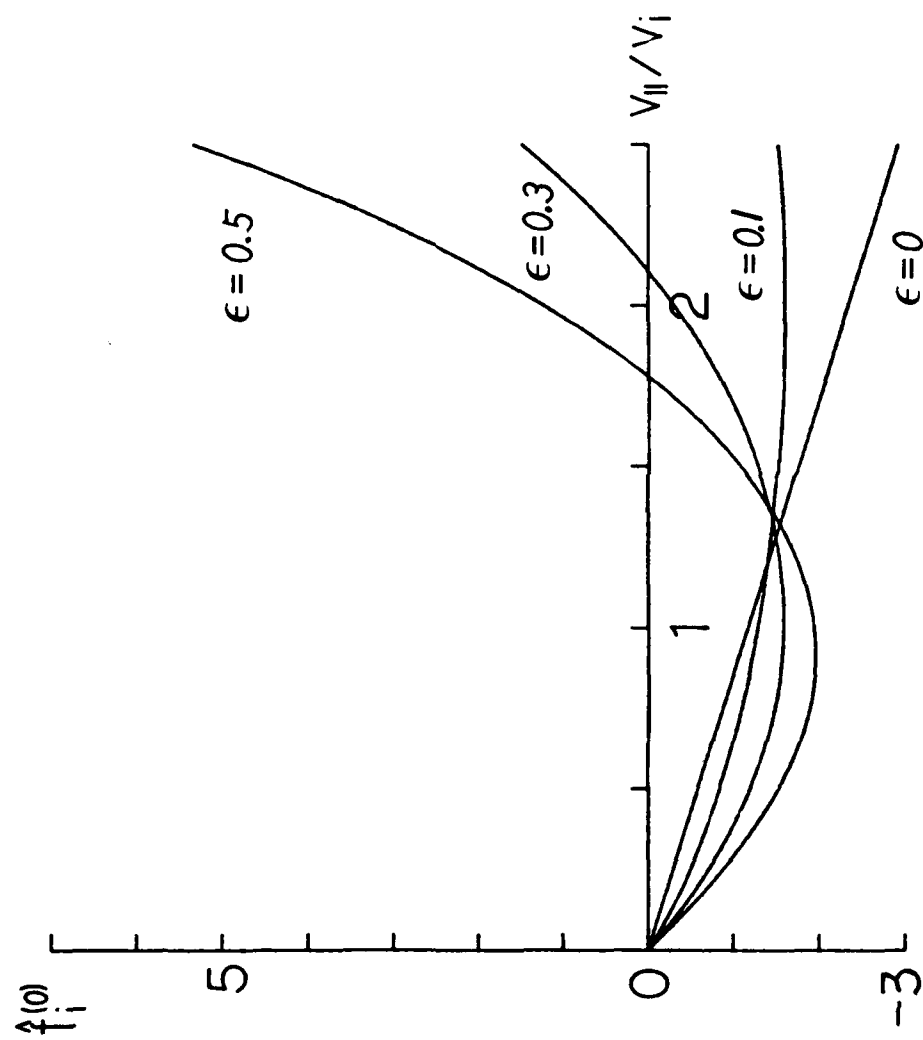
Fig.1

Fig. 2

Fig.3

Fig. 4